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PROGRESS REPORT.

The "*American Mathematical Monthly*" is being received, through the kindness of its publishers, in exchange for our Journal, and numbers of that Journal from the beginning of this year are available for issue on application to the Librarian.

2. The following books have been received as presents to the Library from their publishers—

1. *Bombay University Calendar* 1914, Parts I & II ;
2. *Squaring the Circle*—by Dr. E.W. Hobson, Camb. Univ. Press, 1913, 3*s*. net ;
3. *John Napier*—by Dr. E. W. Hobson, Camb. Univ. Press, 1914, 1*s*. 6*d*. net ;
4. *Complex Integration and Cauchy's Theorem*—by G. N. Watson, (Camb. Math. Tracts, No. 15), 3*s*. net, 1914 ;
5. *Dynamics*—by H. Lamb, Camb. Univ. Press, 1914, 10*s*. 6*d*. net.
6. *A History of Japanese Mathematics*—by D.E. Smith and Yoshio Mikami, The Open Court Publishing Co., 1914, 12*s*. net ;
7. *An Introduction to the Infinitesimal Calculus*—by G. W. Caunt, Oxford Univ. Press, 1914, 12*s*.
8. *Les Coordonnees Intrinseques*—(Theorie et Applications) by L. Braude, (Scientia Series, No. 34), Gauthier-Villars, 1914, 2 frs.
9. *Statics*, Part I—by R. C. Fawdry, G. Bell & Sons, 1914, 2*s*. 6*d*. ;
10. *Arithmetic*—by H. Freeman, G. Bell & Sons, 1914, 2*s*. 6*d*.

POONA,
31st July 1914.

D. D. KAPADIA,
Hony. Joint Secretary.

Statement of Accounts of the Indian Mathematical Society for the half-year ended the 30th June 1914.

Receipts—	Rs.	A.	P.	*Expenditure*—	Rs.	A.	P.
Opening Balance	970	1	9	Books and Journals	268	14	9
Subscriptions from Members	331	8	0	Library	140	4	0
Subscriptions for Journal	80	13	0	Printing Journal	85	13	6
Miscellaneous	0	14	0	Working expenses	75	9	11
				Closing balance*	812	10	7
Total ...	1,383	4	9	Total ...	1,383	4	9

	Rs.	A.	P.
* With the Treasurer	550	15	1
,, Secretary	18	3	0
,, Assistant Secretary	90	5	6
,, ,, Librarian	153	3	0
Total ...	812	10	7

MADRAS,
28th July 1914.

S. NARAYANA AIYAR,
Hon. Treasurer.

On the Bicircular Quartic.

By A. C. L. WILKINSON, M.A., F.R.A.S.

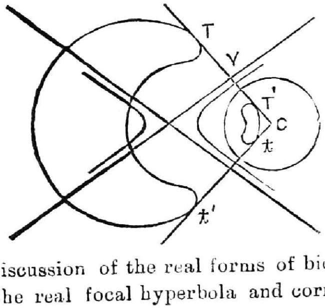
(*Continued from page* 57.)

§ 11. *On the real forms of bicircular quartics.*

In the fig. on page 50 where the circle of inversion cuts the focal ellipse in four real points, there are four real common tangents to the ellipse and circle $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, $\delta\delta'$. As the tangent at P turns round from α to δ the point Q moves from α' through Q to δ', and the point Q′ moves from α' through Q′ to δ', where $CQ.\,CQ'=CY^2$. Similarly the portion of the ellipse $\beta\gamma$ gives the second oval of the figure. Between α and β, or γ and δ the tangents to the ellipse intersect the circle, and the corresponding points Q,Q′ are imaginary.

The four points of intersection of the focal ellipse and circle of inversion are circles of zero radius having double contact with the bicircular and are therefore foci of the bicircular.

Every real bicircular has two real double foci, the real common foci of the focal conics, and four other real single foci. (Basset § 79). The positions of these real foci will be indicated in the following discussion.



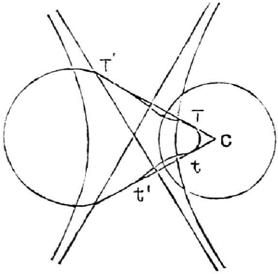
A complete discussion of the real forms of bicirculars can be made by considering the real focal hyperbola and corresponding real circle of inversion. When the circle of inversion does not intersect an asymptote, then the corresponding points of contact of the bitangent are real, as in the fig., where CY is the perpendicular on the asymptote and T, T′ are given by $CY^2-YT^2=$square of radius of the circle of inversion.

When the circle intersects an asymptote the corresponding bitangent is real but its points of contact with the bicircular are imaginary.

To illustrate the genesis of the complete tables that follow, I shall consider the case of the above fig.

Here we have a circle centre C not intersecting either the curve or the asymptotes. There are two branches of the bicircular, 6 real

bitangents with all the points of contact real, 4 real inflexions, no real foci on the focal hyperbola. As the radius of the circle enlarges, its centre remaining fixed, the points T,T′, t,t′ continually approach each other and when the circle touches the hyperbola the two ovals coalesce into a single curve having a node at the point of contact. When the circle cuts the hyperbola in two real points, but not the asymptote, as in the fig. below, the node disappears and we obtain a single oval with two real bitangents and real points of contact, 4 real inflexions, 2 real foci on the focal hyperbola.

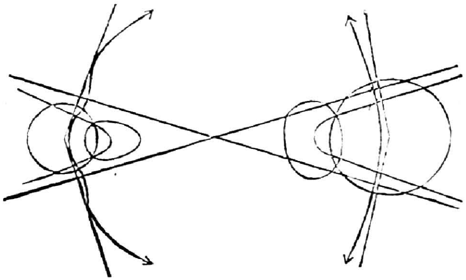


When the circle touches an asymptote the points t,t′ in the fig. coincide and we have a point of undulation on the corresponding bicircular. If the circle cuts one asymptote but not the other the points t,t′ become imaginary, while T,T′ remain real and we obtain a bicircular of one oval, one real bitagent with real points of contact, two real inflexions, two real foci on the focal hyperbola. Continuing to enlarge the circle, the points T,T′ become imaginary, and finally when the the circle touches the other branch of the hyperbola the single oval has shrunk to a conjugate point at the point of contact of the circle and hyperbola. Lastly when the circle intersects both branches of the hyperbola in four real points the bicircular is entirely imaginary.

With this explanation the following table is constructed for the three possible positions of the centre of the circle of inversion with respect to the focal hyperbola:

Notation.—C_0, C_2, C_4 mean none, two, four real intersections of the circle of inversion, in the last case all four must lie on the same branch of the hyperbola. For, when the circle of inversion intersects both branches the corresponding bicircular is imaginary. A_0, A_2, A_4 mean none, two, four real intersections with the asymptotes. Under the column "Bitangents" 5 R. R, 1 R. I, signify five real bitangents with real contacts, one real bitangent with imaginary contacts. Column I

states the real intersections of the circle of inversion with the hyperbol and asympt otes respectively.

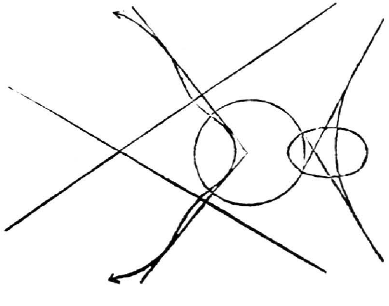


Type I.

The centre of the circle of inversion inside the focal hyperbola.

I	Bicircular.	Bitangents.	Inflexious Real.	Foci on Focal Hyperbola.	Figure.
C_0, A_0	Two exterior ovals.	6. R. R.	4 Two on each oval.	4 Imaginary.	(3)
C_2, A_0	One oval.	2. R. R.	4	2 R.	(4)
C_2, A_2	One oval.	1. R. R. 1. R. I.	2	2 R.	...
C_2, A_4	One oval.	2. R. I.	0	2 R.	...
C_4, A_0	Two interior ovals.	2. R. R.	4	4 R.	(5)
C_4, A_2	Two interior ovals.	1. R. R. 1. R. I.	2	4 R.	...
C_4, A_4	Two interior ovals.	2. R. I.	0	4 R.	(6)

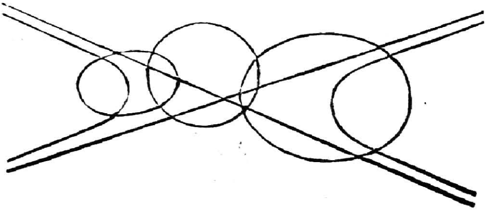
The ovals indicated by the arrows in the figs are very large.

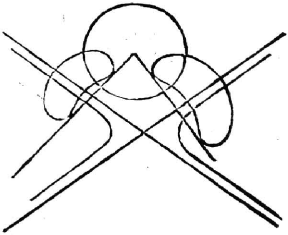


Type II.

Centre of circle of inversion lying between the hyperbola and its asymptotes.

I	Bicircular.	Bitangents.	Inflexions Real.	Foci on focal hyperbola.	Figure.
C_0, A_0	Two exterior ovals.	6 R. R.	4 All on one oval.	4 I	(7)
C_0, A_2	Two exterior ovals.	5 R. R. 1 R. I.	2	4 I	...
C_0, A_4	Two exterior ovals.	4 R. R. 2 R. I.	0	4 I	...
C_2, A_0	One oval.	2 R. R.	4	2 R	...
C_2, A_2	One oval.	1 R. R. 1 R. I.	2	2 R	...
C_2, A_4	One oval.	2 R. I.	0	2 R	...





Type III.

Centre of circle of inversion between the portion of the asymptotes in which the conjugate hyperbola lies.

I	Bicircular.	Bitangents.	Inflexions.	Foci on focal hyperbola.	Figure.
C_0, A_0.	Two exterior ovals.	6 R. R.	4 Nos. on each oval.	4 I.	(8)
C_0, A_2	Two external ovals.	5 R. R. 1 R. I.	2	4 I	...
C_0 A_4	Two external ovals.	4 R. R. 2 R. I.	0	4 I.	(9)
C_2 A_2	One oval.	1 R. R. 1 R. I.	2	2 R.	...
C_2 A_4	One oval.	2 R. I.	0	2 R.	...
C_4 A_2	Two interior ovals.	1 R. R. 1 R. I.	2	4 R.	...
C_4, A_4	Two interior ovals.	2 R. I.	0	4 R.	...

Again since every bicircular can be generated by means of the real focal ellipse and real circle of inversion, we obtain the following classification.

C_0. Two interior ovals with four imaginary foci on the focal ellipse.

C_2. One oval with two real foci on the focal ellipse.

C_4. Two exterior ovals with four real foci on the focal ellipse.

These results, by inspection of figures, will be seen to hold good whether the centre of inversion is inside or outside the ellipse.

The following theorems now follow from the preceeding classification.

1. *When a bicircular consists of two exterior ovals, its four real foci lie on a focal ellipse with a corresponding real circle of inversion.*
2. *When a bicircular consists of two interior ovals, its four real foci lie on a focal hyperbola with a corresponding real circle of inversion.*
3. *When the bicircular consists of a single oval, it has 2 real foci on the focal ellipse with a corresponding real circle of inversion and two real foci on the focal hyperbola with a corresponding real circle of inversion.*

Again reverting to the classification of § 9, we can derive the following results.

4. *When a bicircular can be generated by two real focal ellipses and corresponding real circles of inversion and by one real focal hyperbola and corresponding real circle of inversion, it must consist of two ovals, one interior to the other.*

For were it two external ovals there would be 8 real foci; were it a single oval there would be 2 real foci on each ellipse and 2 real foci on the hyperbola.

5. *When a bicircular can be generated by two real focal hyperbolas and one real focal ellipse with corresponding real circles of inversion, it consists of two ovals mutually exterior to each other.*

6. *When a bicircular can be generated in only two real ways, it consists of a single oval and has two real foci on the real focal ellipse and two real foci on the real focal hyperbola.*

7. *A bicircular quartic whether a single oval or two ovals can have none, two or four real inflexions, but no more.*

§ 12. *The focal properties of the bicircular quartics.*

The following general theorem may be verified. *If from any point on a bicircular quartic tangents are drawn to three circles of the same system having double contact with the bicircular, they are connected by a linear relation.*

Consider the bicircular

$$(x^2+y^2-c)^2-4a^2(x-\alpha)^2-4b^2(y-\beta)^2=0,$$

and three circles

$$x^2+y^2-c-2(x-\alpha)\,a\cos\theta-2(y-\beta)\,b\sin\theta=0.$$

Denote these circles by $r_1^2=0$, $r_2^2=0$, $r_3^2=0$ for values θ_1, θ_2, θ_3 respectively of θ.

It is required to verify that quantities l, m, n can be found so that

$$lr_1+mr_2+nr_3=0$$

is identical with the given quartic.

When completely rationalised the linear relation becomes:

$$l^4r_1^4+m^4r_2^4+n^4r_3^4-2\,l^2m^2r_2^2r_1^2-2\,l^2n^2r_1^2r_3^2-2\,m^2n^2r_2^2r_3^2=0.$$

Equating co-efficients between this equation and the given bicircular, the following six relations must be satisfied:

$$\Sigma l^4-2\,\Sigma l^2m^2=1 \quad \ldots \quad (1)$$

$$\Sigma l^4\cos\theta_1-\Sigma l^2m^2(\cos\theta_1+\cos\theta_2)=0 \quad \ldots \quad (2)$$

$$\Sigma l^4\sin\theta_1-\Sigma l^2m^2(\sin\theta_1+\sin\theta_2)=0 \quad \ldots \quad (3)$$

$$2\,\Sigma l^4\sin\theta_1\cos\theta_1-2\,\Sigma l^2m^2\sin(\theta_1+\theta_2)=0 \quad \ldots \quad (4)$$

$$\Sigma l^4\cos^2\theta_1-2\,\Sigma l^2m^2\cos\theta_1\cos\theta_2+1=0 \quad \ldots \quad (5)$$

$$\Sigma l^4\sin^2\theta_1-2\,\Sigma l^2m^2\sin\theta_1\sin\theta_2+1=0 \quad \ldots \quad (6)$$

From (2) and (3) we deduce

$$\Sigma l^4 e^{i\theta_1} - \Sigma l^2 m^2 (e^{i\theta_1} + e^{i\theta_2}) = 0,$$

which can be written

$$l^2(l^2-m^2-n^2)e^{i\theta_1} + m^2(m^2-n^2-l^2)e^{i\theta_2} + n^2(n^2-l^2-m^2)e^{i\theta_3} = 0,$$

also

$$l^2(l^2-m^2-n^2)e^{-i\theta_1} + m^2(m^2-n^2-l^2)e^{-i\theta_2} + n^2(n^2-l^2-m^2)e^{-i\theta_3} = 0\ ;$$

whence

$$\frac{l^2(l^2-m^2-n^2)}{\sin(\theta_2-\theta_3)} = \frac{m^2(m^2-n^2-l^2)}{\sin(\theta_3-\theta_1)} = \frac{n^2(n^2-l^2-m^2)}{\sin(\theta_1-\theta_2)}$$

By adding the numerators and denominators and taking account of (1), each of these ratios

$$= \frac{1}{-4\sin\frac{\theta_2-\theta_3}{2}\sin\frac{\theta_3-\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}}.$$

Subtracting (5) and (6), we have

$$\Sigma l^4 \cos 2\theta_1 - 2\,\Sigma l^2 m^2 \cos(\theta_1+\theta_2) = 0.$$

Combining this equation with (4), we obtain

$$\Sigma l^4 e^{2i\theta_1} - 2\,\Sigma l^2 m^2 e^{i(\theta_1+\theta_2)} = 0.$$

We thus obtain the relations

$$\left.\begin{aligned} &l e^{\frac{1}{2}i\theta_1} + m e^{\frac{1}{2}i\theta_2} + n e^{\frac{1}{2}i\theta_3} = 0, \\ &l e^{-\frac{1}{2}i\theta_1} + m e^{-\frac{1}{2}i\theta_2} + n e^{-\frac{1}{2}i\theta_3} = 0, \end{aligned}\right\} \quad \ldots \quad \ldots \quad (7)$$

$$\left.\begin{aligned} &l e^{\frac{1}{2}i\theta_1} + m e^{\frac{1}{2}\theta_2} + n e^{\frac{1}{2}i\theta_3} = 0, \\ &l e^{-\frac{1}{2}i\theta_1} + m e^{-\frac{1}{2}i\theta_2} - n e^{-\frac{1}{2}i\theta_3} = 0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (8)$$

Consider the system (7); solving we get

$$\frac{l}{\sin\frac{\theta_2-\theta_3}{2}} = \frac{m}{\sin\frac{\theta_3-\theta_1}{2}} = \frac{n}{\sin\frac{\theta_1-\theta_2}{2}} = k.$$

Hence

$$l^2(l^2-m^2-n^2) = k^4 \sin^2\frac{\theta_2-\theta_3}{2}\left\{\sin^2\frac{\theta_2-\theta_3}{2} - \sin^2\frac{\theta_3-\theta_1}{2} - \sin^2\frac{\theta_1-\theta_2}{2}\right\}$$

$$= k^4 \sin(\theta_2-\theta_3)\sin\frac{\theta_2-\theta_3}{2}\sin\frac{\theta_3-\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}.$$

Comparing this with our previous expression, we have

$$k^4 = -\frac{1}{4\sin^2\frac{\theta_2-\theta_3}{2}\sin^2\frac{\theta_3-\theta_1}{2}\sin^2\frac{\theta_1-\theta_2}{2}} \quad \ldots \quad \ldots \quad (10)$$

The values of l, m, n given by (9) and (10) thus render consistent the equations (1), (2), (3), (4) and the difference of (5) and (6). We have finally to show that these values satisfy also the sum of (5) and (6); or that

$$\Sigma l^4 - 2\,\Sigma l^2 m^2 \cos(\theta_1-\theta_2)+2=0 \quad \ldots \quad \ldots \quad (11)$$

Multiply the two equations of (7) together, therefore

$$l^2+m^2+n^2=-2\left(lm\cos\frac{\theta_1-\theta_2}{2}+mn\cos\frac{\theta_2-\theta_3}{2}+nl\cos\frac{\theta_3-\theta_1}{2}\right)$$

Squaring therefore

$$\Sigma l^4+2\,\Sigma l^2m^2=2\,\Sigma l^2m^2\,\{1+\cos(\theta_1-\theta_2)\}$$
$$+8\,lmn\Sigma l\cos\frac{\theta_1-\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}.$$

Now

$$\Sigma\sin\frac{\theta_2-\theta_3}{2}\cos\frac{\theta_1-\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}=\sin\frac{\theta_2-\theta_3}{2}\sin\frac{\theta_3-\theta_1}{2}\sin\frac{\theta_1-\theta_2}{2}.$$

Thus

$$8\,lmn\,\Sigma l\cos\frac{\theta_1-\theta_2}{2}\cos\frac{\theta_3-\theta_1}{2}=8\,k^4\sin^2\frac{\theta_2-\theta_3}{2}\sin^2\frac{\theta_3-\theta_1}{2}\sin^2\frac{\theta_1-\theta_2}{2}$$
$$=-2.$$

Hence equation (11) is satisfied.

Considerations of reality of the quantities θ_1, θ_2, θ_3, l, m, n have not so far been taken into account. It would appear at first sight from (10) that all three quantities are not real together; had we however written equations (1), (5), (6) in the form

$$\Sigma l^4-2\,\Sigma l^2m^2=-1,$$
$$\Sigma l^4\cos^2\theta_1-2\,\Sigma l^2m^2\cos\theta_1\cos\theta_2=1,$$
$$\Sigma l^4\sin^2\theta_1-2\,\Sigma l^2m^2\sin\theta_1\sin\theta_2=1:$$

we should have obtained

$$k^4=\frac{1}{4\sin^2\frac{\theta_2-\theta_3}{2}\sin^2\frac{\theta_3-\theta_1}{2}\sin^2\frac{\theta_1-\theta_2}{2}}.$$

It remains to consider whether such an equation as (8) is possible. Solving (8) we obtain

$$\frac{l}{-\cos\frac{\theta_2-\theta_3}{2}}=\frac{m}{\cos\frac{\theta_3-\theta_1}{2}}=\frac{n}{i\sin\frac{\theta_1-\theta_2}{2}}.$$

The ratio

$$\frac{l^2(l^2-m^2-n^2)}{m^2(m^2-n^2-l^2)}\text{ will be found to be }\frac{\cos^3\frac{\theta_2-\theta_3}{2}\sin\frac{\theta_3-\theta_1}{2}}{\cos^3\frac{\theta_3-\theta_1}{2}\sin\frac{\theta_2-\theta_3}{2}},$$

which is inconsistent with the ratio $\frac{\sin(\theta_2-\theta_3)}{\sin(\theta_3-\theta_1)}$. Thus only equation (7) holds good.

Hence the equation

$$r_1 \sin\frac{\theta_2-\theta_3}{2} \pm r_2 \sin\frac{\theta_3-\theta_1}{2} \pm r_3 \sin\frac{\theta_1-\theta_2}{2}=0$$

when rationalised is identical with the given bicircular quartic.

§ 13. *Application to the real foci of a bicircular.*

Case I: when the bicircular consists of two exterior ovals so that the four real foci lie on an ellipse.

Here θ_1, θ_2, θ_3 are real for the point circles or foci and the relation between any three foci and their distances to any point on the bicircular is

$$r_1 \sin\frac{\theta_2-\theta_3}{2} \pm r_2 \sin\frac{\theta_3-\theta_1}{2} \pm r_3 \sin\frac{\theta_1-\theta_2}{2}=0.$$

Case II: when the bicircular consists of two interior ovals, so that the four foci lie on a hyperbola. The four foci all lie on the same branch of the hyperbola and their co-ordinates may be taken to be $(a \cosh \theta_1,\ b \sinh \theta_1)$ $(a \cosh \theta_2,\ b \sinh \theta_2)$ $(a \cosh \theta_3,\ b \sinh \theta_3)$ if they lie on the branch of the hyperbola for which the abscissa is positive.

The relation between the foci and the bicircular is now.

$$r_1 \sinh\frac{\theta_2-\theta_3}{2} \pm r_2 \sinh\frac{\theta_3-\theta_1}{2} \pm r_3 \sinh\frac{\theta_1-\theta_2}{2}=0.$$

Case III: when the bicircular consists of a single oval so that two real foci lie on a focal ellipse and two on a focal hyperbola.

As the circles are not now of the same system no relation of the linear form holds good.

§ 14. *Coversely*, if a bicircular is given by the relation $lr_1+mr_2+nr_3=0$, where r_1, r_2, r_3, are the distances from three fixed points, these points lie on one of the circles of inversion of the bicircular.

Take the circle on which the points lie as

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2.$$

Denote the three points on this circle by

$(\alpha+\gamma \cos \theta,\ \beta+\gamma \sin \theta)$ for values θ_1, θ_2, θ_3 of θ.

Rationalising the expression

$$\Sigma l \left\{ (x-\alpha-\gamma \cos \theta_1)^2+(y-\beta-\gamma \sin \theta_1)^2 \right\}^{\frac{1}{2}}=0,$$

we obtain

$$\Sigma l^4[(x-\alpha)^2+(y-\beta)^2-2\gamma(x-\alpha) \cos \theta_1-2\gamma(y-\beta) \sin \theta_1+\gamma^2]=0.$$

Choose l, m, n so that $\Sigma l^4 - 2\Sigma l^2m^2 = 1$. Then denoting the expressions (2), (3), (5), (6), (4) of § 12 by A, B, C+1, D+1, E we may write the above bicircular in the form

$$[(x-\alpha)^2+(y-\beta)^2+\gamma^2]^2 - 4\{(x-\alpha)^2+(y-\beta^2+\gamma^2\}$$
$$\{A\gamma(x-\alpha)+B\gamma(y-\beta)\}$$
$$+4\gamma^2(x-\alpha)^2+4\gamma^2 D(y-\beta)^2+8\gamma^2 E(x-\alpha)(y-\beta)=0;$$

or $[(x-\alpha)^2+(y-\beta)^2+\gamma^2-2A\gamma(x-\alpha)-2B\gamma(y-\beta)]^2$

$$=4\gamma^2(A^2-C)(x-\alpha)^2+4\gamma^2(B^2-D)(y-\beta)^2$$
$$+8\gamma^2(AB-E)(x-\alpha)(y-\beta).$$

Now the origin and the direction of the axes are at our disposal. Take $A\gamma+\alpha=0$, $B\gamma+\beta=0$ determining α and β, and choose the direction of the axes so that $AB-E=0$. The bicircular then becomes

$$(x^2+y^2-\alpha^2-\beta^2+\gamma^2)^2=4\gamma^2(A^2-C)(x-\alpha)^2+4\gamma^2(B^2-D)(y-\beta)^2$$

of which the circle of inversion is

$$(x-\alpha)^2+(y-\beta)^2=\gamma^2.$$

§ 15. In certain cases the preceding general theorems degenerate. Suppose $\beta=0$. The corresponding bicirculars are symmetrical. From § 7 we see that one root of the cubic in λ is given by $\lambda+b=0$ and

$$\frac{\beta^2}{b+\lambda}=\frac{c-b-\dfrac{a\alpha^2}{a-b}}{b} \quad \text{in the limit.}$$

The corresponding focal conic degenerates to $y=0$, the value of β' is infinite and the corresponding circle of inversion also degenerates to $y=0$. In this case four of the foci are collinear.

The bicircular now assumes a simple form in § 13.

Consider the focal conic $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ for $b=0$, if three of the foci are at distances u_1, u_2, u_3 from the centre of the focal conics, we have

$$u_1=a\cosh\theta_1,\ u_2=a\cosh\theta_2,\ u_3=a\cosh\theta_3 \text{ and}$$

$$\sqrt{2}\sinh\frac{\theta_1-\theta_2}{2}=\sqrt{\cosh(\theta_1-\theta_2)-1}$$
$$=\frac{1}{a}\sqrt{u_1u_2-a^2-\sqrt{(u_1^2-a^2)(u_2^2-a^2)}}$$
$$=\frac{1}{2a}\left\{\sqrt{(u_1+a)(u_2-a)}-\sqrt{(u_1-a)(u_2+a)}\right\}.$$

If S,S′ are the double foci of the bicircular and F_1, F_2, F_3 three collinear foci, we obtain for the bicircular the equation

$$\sum\left\{\sqrt{SF_1.S'F_2}-\sqrt{S'F_1.SF_2}\right\}r_3=0.$$

Consider further the case of degeneration when the bicircular becomes a cartesian.

Then $a=b$, thus SS′ coincide with O the centre of the focal circle.

The ratios of $\sinh\frac{\theta_1-\theta_2}{2}:\sinh\frac{\theta_2-\theta_3}{2}:\sinh\frac{\theta_3-\theta_1}{2}$ have now to be evaluated for $a=0$.

We have

$$\sqrt{2}\sinh\frac{\theta_1-\theta_2}{2}=\frac{1}{a}\sqrt{u_1u_2-a^2-\sqrt{(u_1^2-a^2)(u_2^2-a^2)}}$$

$$=\frac{1}{-}\sqrt{u_1u_2-a^2-u_1u_2\left(1-\frac{1}{2}\frac{a^2}{u_1^2}-\frac{1}{2}\frac{a^2}{u_2^2}\right)}.$$

$$=\frac{1}{\sqrt{2}}\left\{\sqrt{\frac{u_1}{u_2}}-\sqrt{\frac{u_2}{u_1}}\right\}$$

Hence if u_1, u_2, u_3 are the distances of the foci of a cartesian from the triple focus, the equation to the cartesian is

$$(u_1-u_2)\sqrt{u_3r_3}\pm(u_2-u_3)\sqrt{u_1r_1}\pm(u_3-u_1)\sqrt{u_2r_2}=0.$$

More generally the above theorem holds good if r_1, r_2, r_3 are the lengths of the tangents from any point on a cartesian to three circles having double contact with the cartesians whose centres lie on the axis of the cartesian at distances u_1, u_2, u_3 from the triple focus.

§ 16. *The fundamental property of a cartesian.*

For a cartesian the focal conics degenerate into circles and in particular one focal hyperbola and the corresponding circle of inversion degenerate into the axis of the cartesian. The equation of a cartesian can be written

$$(x^2+y^2-c)^2=4a^2(x-\alpha)^2+4a^2y^2=0,$$

where the equations of the focal circle and circle of inversion are

$$x^2+y^2=a^2,\ x^2-2\alpha x+y^2+c=0.$$

The cubic in λ becomes

$$(a^2+\lambda)\left\{(a^2+\lambda)(\lambda-c)+a^2\alpha^2\right\}=0.$$

The factor $a^2+\lambda=0$ corresponds to the case where the focal circle and circle of inversion coincide with the axis of x.

The quadratic in λ is

$$\lambda^2+\lambda(a^2-c)+a^2(\lambda^2-c)=0.$$

Now if α^2-c is positive, the circle $x^2+y^2-2\alpha x+c=0$ is real; if α^2-c is negative the roots of the quadratic in λ are real and one is positive and one negative.

From § 9 we now draw the following conclusions:

There are two types of cartesians:

(1) Those for which the cartesian can be generated by means of two real focal circles and corresponding real circles of inversion. These are the ovals of Descartes.

(2) Those for which the cartesian can be generated in only one way by means of a real focal circle and corresponding real circle of inversion.

We may notice that the focal circle and circle of inversion intersect in real points if $2\alpha x-a^2-c=0$ meets $x^2+y^2=a^2$ in real points; that is, if

$$(a^2+c)^2<4\alpha^2a^2,$$

which is the same as the condition that the quadratic in λ should have imaginary roots.

Thus for the first type of cartesians the focal circle and circle of inversion do not intersect in real points and the curve consists of two ovals one within the other. In the second type of cartesians the circles intersect in real points and the curve consists of a single oval.

§ 17. Some of the properties of the focal circle which degenerates to the axis of the cartesian may be obtained as follows:

The cartesian whose equation is

$$(x^2+y^2-c)^2=4a^2(x-\alpha)^2+4a^2y^2$$

can be written

$$(x^2+y^2-c-2a^2)^2+8a^2\alpha x-4a^2(a^2+\alpha^2+c)=0.$$

The straight line $2\alpha x-a^2-\alpha^2-c=0$ is the only bitangent to the cartesian.

From any point on the axis of x a circle can be drawn having double contact with the cartesian. For the circle

$$(x-u)^2+y^2=\rho^2$$

can be written $\qquad x^2+y^2-c-2a^2=2ux-c-2a^2+\rho^2-u^2,$

and the condition for double contact is that the equation

$$(2ux-c-2a^2+\rho^2-u^2)^2+8a^2\alpha x-4a^2(a^2+\alpha^2+c)=0$$

may have equal roots.

Now $\qquad (x-b)^2+cx+d=(x+k)^2,$

if $\qquad cx+d=(2x+k-b)(k+b),$

whence $\qquad 2(k+b)=c,\ k-l=2d/c,$

and $\qquad 4b=c-4d/e.$

Thus the condition required is that

$$\frac{c+2a^2-\rho^2+u^2}{u}=\frac{a^2\alpha}{u^2}+\frac{a^2+\alpha^2+c}{\alpha}.$$

This gives ρ^2 for any value of u.

The foci on the axis of x are given by $\rho^2=0$: whence

$$u(u^2+c+2a^2)-a^2\alpha-\frac{a^2+\alpha^2+c}{\alpha}u^2=0;$$

a cubic in u of which one root is $u=\alpha$, and the other roots are given by the quadratic

$$u^2-\frac{u}{\alpha}(c+a^2)+a^2=0.$$

The roots of this are real if $(c+a^2)^2>4a^2\alpha^2$, which is the same as the condition that the roots of the quadratic in λ in § 16 may be real.

Hence when a cartesian consists of two ovals, it has 3 real foci on the axis, when it consists of a single oval it has any one real focus on the axis.

From § 15 if u_1, u_2, u_3 are the distances from the triple focus of any three circles, centres on the axis, having double contact with the cartesian, the equation to the cartesian can be written

$$(u_1-u_2)\sqrt{u_3}\,r_3\pm(u_2-u_3)\sqrt{u_1}\,r_1\pm(u_3-u_1)\sqrt{u_2}\,r_2=0.$$

For the three foci the result of substituting $u_1=\alpha$, u_2 and u_3 the roots of $u^2-\frac{u}{\alpha}(c+a^2)+a^2=0$, gives the cartesian referred to its three collinear foci.

Again, for $u=0$, $\rho=\infty$, and thus the straight line at infinity is a circle having double contact with the cartesian, as is geometrically obvious.

Taking as three circles having double contact with the cartesian, two of the foci and the line at infinity, we have for $u_3=0$, $\rho=\infty$, the limit of $\sqrt{u_3}\,r_3=\sqrt{u_3}\,(x^2+y^2-\rho^2)=a\sqrt{\alpha}$.

Thus the cartesian may be expressed in any one of the four forms

$$(u_1-u_2)\sqrt{\alpha}\,r_3\pm(u_2-\alpha)\sqrt{u_1}\,r_1\pm(\alpha-u_1)\sqrt{u_2}\,r_2=0 \quad \ldots \quad (1)$$

$$a\sqrt{\alpha}\,(u_1-u_2)\pm u_2\sqrt{u_1}\,r_1\pm u_1\sqrt{u_2}\,r_2=0 \quad \ldots \quad (2)$$

$$a\sqrt{\alpha}\,(u_1-\alpha)\pm\alpha\sqrt{u_1}\,r_1\pm u_1\sqrt{\alpha}\,r_3=0 \quad \ldots \quad (3)$$

$$a\sqrt{\alpha}\,(u_2-\alpha)\pm\alpha\sqrt{u_2}\,r_2\pm u_2\sqrt{\alpha}\,r_3=0 \quad \ldots \quad (4)$$

where the origin is the triple focus; α, u_1, u_2 the distances of the three foci from the origin; r_3, r_1, r_2 the distances of a point of the cartesian from these foci respectively.

These relations are not independent; for (3) and (4) can be deduced from (1) and (2) by elimination of r_2, r_1 respectively.

§ 18. In conclusion I shall identify the results just obtained with those given in Basset §§ 259 to 277.

The cartesian may be written

$$\{x+\alpha)^2+y^2-c\}^2=4a^2(x^2+y^2)$$

or $(r^2+2\alpha r\cos\theta+\alpha^2-c)^2=4a^2r^2$, which is identical with Basset's $r+mr_1=k$,

or $$\left\{r^2+\frac{2m^2\lambda}{1-m^2}r\cos\theta+\frac{k^2-m^2\lambda^2}{1-m^2}\right\}^2=\frac{4k^2}{(1-m^2)^2}r^2,$$

where λ, k are written for Basseet's c_1, a_3, if

$$\alpha=\frac{m^2\lambda}{1-m^2},\ \alpha^2-c=\frac{k^2-m^2\lambda^2}{1-m^2},\ a^2=\frac{k^2}{(1-m^2)^2}.$$

Hence $1-m^2=\dfrac{\lambda}{\lambda+\alpha}$, $a^2-\dfrac{\alpha^2-c}{1-m}=\dfrac{m^2\lambda^2}{(1-m^2)^2}=\dfrac{\lambda\alpha}{1-m^2}$, giving for λ the equation

$$\frac{\lambda}{\lambda+\alpha}\ a^2-(\alpha^2-c)=\lambda\alpha.$$

Writing $\lambda+\alpha=u$, this becomes

$$(u-\alpha)\ a^2-uc=u^2\alpha$$

the quadratic of § 17 giving the foci on the axis of the cartesian other than $u=\alpha$.

Now $u_3=\dfrac{m^2\lambda}{1-m^2}$, $u_1=\lambda+\dfrac{m^2\lambda}{1-m^2}$, $u_2=\lambda'+\dfrac{m^2\lambda}{1-m^2}$, where λ, λ' are the roots of the equation in λ.

The equation (1) of § 17 becomes

$$(\lambda-\lambda')\frac{m\sqrt{\lambda}}{\sqrt{1-m^2}}r_3\pm\lambda'\sqrt{\lambda+\frac{m^2\lambda}{1-m^2}}r_1\pm\lambda\sqrt{\lambda'+\frac{m^2\lambda}{1-m^2}}r_2=0$$

or $$m(\lambda-\lambda')r_3\pm\lambda' r_1\pm\sqrt{m^2\lambda^2+\lambda\lambda'(1-m^2)}\,r_2=0,$$

which agrees with the result of eliminating the constants between the equations (7) and (8) of Bassett § 262 and substituting for a^2 the expression $m^2\lambda^2+\lambda\lambda'(1-m^2)$ from § 261.

Equation (3) of § 17 becomes

$$\frac{k}{1-m^2}\cdot\frac{m\sqrt{\lambda}}{1-m^2}\lambda\pm\frac{m^2\lambda}{1-m^2}\frac{\sqrt{\lambda}}{\sqrt{1-m^2}}r_1\pm\frac{\lambda}{1-m^2}\frac{m\sqrt{\lambda}}{\sqrt{1-m^2}}r_3=0,$$

or $$k\pm mr_1\pm r_3=0,$$

agreeing with equation (1) of Bassett § 261.

The other equations can be similarly identified.

Stereographic Projection.

BY M. T. NARANIENGAR.

[Stereographic projection is usually defined as the projection of a spheical curve on the plane of a great circle, the centre of projection being the pole of the great circle. We shall, however, define it as the projection on a tangent plane, the centre of projection being the point diametrically opposite to the point of contact.

Thus defined the stereographic projection of any curve is simply its inverse in space with respect to the origin, the radius of inversion being the diameter of the fundamental sphere].

1. *The stereographic projection of any spherical circle is a circle.*

This follows from the well-known property that the inverse of a circle in space is a circle.

Cor. If the spherical circle passes through the centre of projection the inverse is a straight line. Hence the stereographic projection of such a circle is a straight line.

2. *The angle of intersection of any two spherical curves is unaltered by stereographic projection.*

This is simply a property of inversion.

Hence orthogonal circles on the sphere project into orthogonal circles in a plane, and so on.

3. *The stereographic projection of a diametral spherical triangle ABC $(C=A+B)$ on the tangent plane at A.*

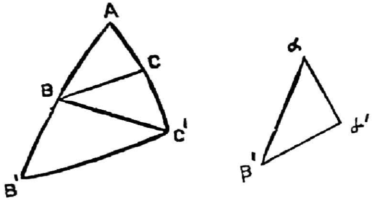
Denoting the projections of A, B, C by the Greek letters α, β, γ $(\alpha \equiv A)$ we see that the sides AB, AC, BC of the spherical triangle project into the right lines $\alpha\beta$, $\alpha\gamma$ and an arc $\beta\gamma$. Further, the cicumcircle ABC becomes the circle $\alpha\beta\gamma$; and as AB cuts at right angles the circle ABC, the straight line $\alpha\beta$ must also cut at right angles the circle $\alpha\beta\gamma$. Hence, $\alpha\beta$ is a diameter of the circle $\alpha\beta\gamma$. That is, the plane triangle $\alpha\beta\gamma$ is *right-angled* at γ, having the angle α equal to the angle A.

Cor. The centre of the arc $\beta\gamma$ is ω, such that the angle $\omega\beta\gamma = \omega\gamma\beta = \pi - C$.

This follows from the property that the arc $\beta\gamma$ makes angles B and C with the right lines $\alpha\beta$, $\alpha\gamma$ respectively.

4. *Properties of a spherical right-angled triangle.*

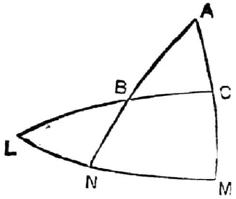
Let ABC be a spherical triangle right-angled at C. Produce AB,



AC to B′, C′ so that BB′ = AB, CC′ = AC. Then evidently △ABC ≡ C′BC and AB′C is a diametral triangle having C′ = A + B′. Therefore by § 3

$$\cos A = \cos \alpha = \frac{\alpha\gamma'}{\alpha\beta'} = \frac{\tan AOC'}{\tan AOB'} = \frac{\tan \frac{1}{2} AC'}{\tan \frac{1}{2} AB'} = \frac{\tan AC}{\tan AB} = \frac{\tan b}{\tan c}, \quad \ldots(1)$$

O being the point diametrically opposite to A.



Next, produce the sides of ABC to meet the polar great circle of A at L, M, N.

Then in the right-angled triangle LBN

$$\cos L = \frac{\tan LN}{\tan LB} \text{ by (1)}$$

But $L = \frac{\pi}{2} - b$, $LN = \frac{\pi}{2} - A$, $LB = \frac{\pi}{2} - a$, since $LC = LM = 90°$.

Hence $\sin b = \cot A . \tan a$.

That is $$\tan A = \frac{\tan a}{\sin b} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Further from the triangles BLN, BAC,

$$\tan B = \frac{\tan LN}{\sin BN} = \frac{\tan AC}{\sin BC}.$$

$$\therefore \qquad \frac{\cot A}{\cos c}=\frac{\tan b}{\sin a}.$$

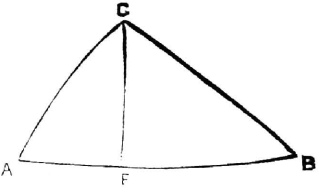
$$\therefore \qquad \cot A=\cos c \tan b \operatorname{cosec} a.$$

Hence, by (2), $\cos c \tan b \operatorname{cosec} a=\cot a \sin b$.

Thus $$\cos c=\cos a \cos b \qquad \dots \qquad \dots \qquad (3)$$

Using (1), (2) and (3) it is quite easy to deduce all the well-known properties of the right-angled triangle.

5. *Properties of the general spherical triangle.*



These can be made to depend upon the properties of the right-angled triangle established in the previous article. Thus to prove the formula for cos A, we may proceed as follows :

$$\cos A=\tan x \cot b,$$

where x stands for AF, CF being $\perp^r$ to AB.

But $$\cos CF=\frac{\cos b}{\cos x}=\frac{\cos a}{\cos (c-x)}.$$

$$\therefore \qquad \frac{\cos (c-x)}{\cos x}=\frac{\cos a}{\cos b}.$$

$$\therefore \qquad \cos c+\sin c \tan x=\frac{\cos a}{\cos b}.$$

$$\therefore \qquad \tan x=\frac{\cos a-\cos b \cos c}{\sin c \cos b}.$$

Hence

$$\cos A=\tan x \cot b=\frac{\cos a-\cos b \cos c}{\sin b \sin c}.$$

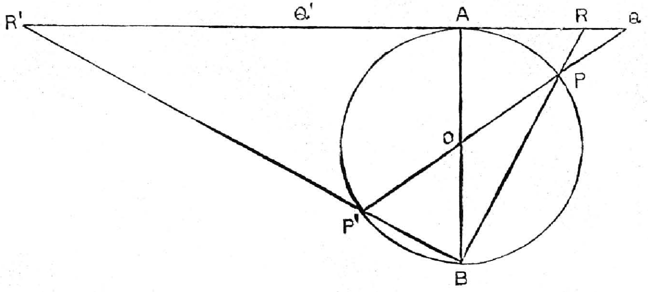
6. *Analytical Investigation.*

Consider the central and stereographic projections of a point P of a sphere on the tangent plane at A. These are Q & R such that ARQ is a straight line. Further, if the angle ABP$=\lambda$,

$$AQ=r=a \tan 2\lambda, \text{ and } AR=r'=2a \tan \lambda.$$

Hence, Q, R are connected by the relation $r=4a^2 r' (4a^2-r'^2)$.

Thus, when the locus of Q is known, that of R is raddily found an is seen to be identical with that of R′, where AR. AR′ $=4a^2$.



The relation between Q & R is geometrically represented as follows :—

$$\frac{4a^2}{r}=\frac{4a^2}{r'}-r'.$$

i.e. $$\frac{4a^2}{AQ}=\frac{4a^2}{AR}-AR=AR'-AR=2\ AQ'$$

where Q′ is the mid-point of RR′.

That is, $AQ\ AQ'=2a^2$.

In other words, Q and Q′ describe inverse curves.

Hence, as P describes a spherical curve, we have the following properties among the plane projections connected therewith :—

(1) Q describes the plane projection of the spherical curve.

(2) R, R′ describe the stereographic projection which is such that $AR.AR'=4a^2$.

Thus the stereographic projection is its own inverse with respect to A, the constant of inversion being $2a\sqrt{-1}$.

(3) The locus of the middle point of RR′ is the inverse of Q with respect to a circle of radius $a\sqrt{-2}$ and centre A.

7. Suppose, the locus of Q is a conic, which may be written, (with the usual notation),

$$u_2+u_1+u_0=0,$$

or $$r^2f_2(\theta)+rf_1(\theta)+u_0=0.$$

Then the locus of R is obtained by substituting for r, $4a^2r'/(4a^2-r'^2)$ in the above.

The result of the substitution is found to be

$$16a^4r'^2f_2(\theta)+4a^2r'f_1(\theta)(4a^2-r'^2)+u_0(4a^2-r'^2)^2=0,$$

or $$16a^4u_2+4a^2u_1(4a^2-r^2)+u_0(4a^2-r^2)^2=0,$$
if we drop the accents.

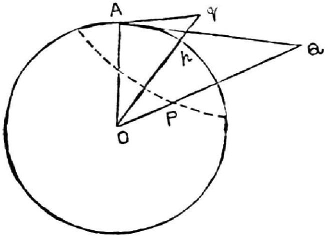
This is a bicircular quartic having the origin for a centre of inversion. (*Vide* : Basset, § 200.)

Casey's method of generation of the quartic therefore shows that Q′ is the foot of the perpendicular from A on a tangent to the focal conic of the quartic traced out by R, R′. In other words, the focal conic is the negative pedal of the locus of Q′, and therefore the locus of Q is the polar reciprocal of the focal conic.

Now, when the locus of Q is a conic, P generally describes a sphero-conic. Hence we have the theorem :

The stereographic projection of a sphero-conic is a bicircular quartic having for a centre of inversion the point diametrically opposite to the centre of projection, whose focal conic is the polar reciprocal of the central projection of the sphero-conic.

In the particular case in which P describes a spherical circle, the stereographic projection degenerates into two circles, and the point A is a centre of inversion for these, being the centre of inverse similitude of the two circles.



Thus, if β be the spherical radius of the circle described by P and p its pole, we have
$$\cos\beta=\cos 2\lambda\cos\alpha+\sin 2\lambda\sin\alpha\cos\theta,$$
where $\alpha=\text{AO}p$, $\theta=\text{QA}q$; and
$$r=a\tan 2\lambda,\ r'=2a\tan\lambda.$$

Hence the locus of R, R′ is
$$\cos\beta\left(+\frac{r'^2}{4a^2}\right)=\pm\left\{\left(1-\frac{r'^2}{4a^2}\right)\cos\alpha+\frac{r'}{a}\sin\alpha\cos\theta\right\},$$
which denotes two inverse circles.

SHORT NOTES.

A card trick.

The explanation of the card trick from the '*Scientific American*' given on p. 15 of Vol. V of the Journal, is as follows:

(Diamonds)	1	2	3	4	5	6	7	8	9	10
(Hearts)	10	1	2	3	4	5	6	7	8	9
(Spades)	9	10	1	2	3	4	5	6	7	8
(Clubs)	8	9	10	1	2	3	4	5	6	7

Denote the position of any card in the above arrangement by (x, y) where x, y are the orders of the column and row to which it belongs counting from the left-hand column and top row respectively. Then it is easily seen that the value of the card is $(x-y+1)$ if this is positive, and $(x-y+1)+10$ otherwise.

In the re-arrangement of the cards, the order of the card (x, y) from 8 of clubs is

$$4(x-1)+(5-y)=4(x-y+1)+3(y-1).$$

Now $(x-y+1)$ is the value of the card if it is positive; and $y-1=0, 1, 2$ and 3 for diamonds, hearts, spades and clubs respectively.

Hence the rule given.

If $(x-y+1)$ is not positive the rule will give a result in excess of the required number by 40. Hence we should subtract 40, in this case.

R. VYTHYNATHASWAMY.

On Question 155.

[*Q.* 155. (T. Rajarama Rao, B.A., B.L.,) :—BE is the radical axis of the circles ABC and DEF. On EB produced take any point P and draw PAC parallel to the tangent drawn to the circle DEF at the point B cutting the circle BAC at A, C. Similarly, draw PDF parallel to the tangent at B to the circle BAC cutting the circle DEF at D and F. Join AD and CF, and produce them to meet at Q. Shew that PQ is parallel to the line drawn through B such that the intercepts made on the line by the two circles are equal.]

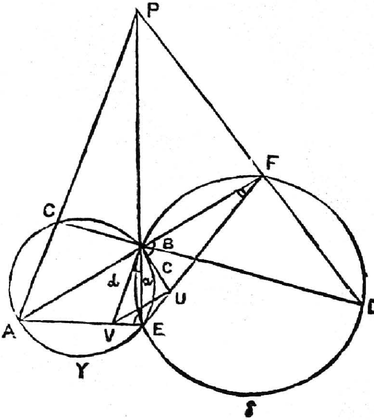
By limiting the position of P to the parts of the radical axis outside the circles, the proposer has been able to present as complete what is in fact only a trivial deduction from two interesting theorems. There are no directions in the enunciation for distinguishing between the points A and C or between the points D and F. The quadrangle ACFD has two harmonic points other than P, and it is only by the position of P that our attention is focussed on the one which has the required property. When we look for the neglected point, we soon find a series of results in which the suggested proposition takes a natural place, for this point is the point B itself. All these theorems are particular cases of theorems on the chords of intersection of any two conics ; the analytical treatment of the general case is the simplest way of deducing the results, but we give also geometrical investigations of both the general case and the particular.

1. Two circles γ, δ intersect in B, E and a line through B meets them again in points A, F ; if the tangent c to γ at B cuts EF at U and the tangent d to δ at B cuts EA at V, then UV is parallel to AF.

Since $B\hat{E}A = U\hat{B}F$ and $V\hat{B}E = B\hat{F}U$, we have BV/VE = FU/UB similarly AV/VB = BU/UE, and the proposition follows on multiplying the ratios.

2. With the same notation as in 1, a line through A parallel to the tangent d, and a line through F parallel to the tangent c intersect on the radical axis BE.

For if the first of these lines meets BE in P, PB/BE = AV/VE = FU/UE and therefore PF is the second line.

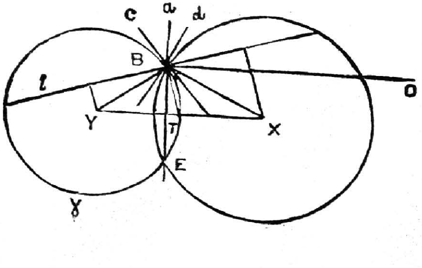


3. Through any point P on the radical axis *a* of two circles γ, δ of which B is one point of intersection are drawn lines parallel to the tangents *c*, *d* at B to γ, δ respectively; if the first of these meets δ in the points D, F and the second meets γ in the points A, C, then B is one of the harmonic points of the quadrangle ACDF.

For if AB, CB meet δ in W, Z the four lines WP, ZP, DP, FP are all parallel to the tangent *c*, and therefore all coincide, so that since W, Z are distinct points they the same pair of points as D, F.

4. Through a point of intersection B of two circles γ, δ are drawn two pairs of lines, the radical axis *a* and the line *l* on which the intercepts made by the two circles are equal forming one pair, and the tangents *c*, *d* to the two circles, forming the other; these pairs are harmonic with respect to each other.

Let Y, X be the centres of γ, δ, let T be the middle point of YX, and let O be any point in the line through B parallel to the line of centres YX. Then the lines *a*, *l*, *c*. *d* are at right angles respectively to BO, BT, BY, BX, and the pair BO, BT is harmonic with respect to the pair BY, BX.



5. **Two circles γ, δ have β for a common point; the chords AC of γ and DF of δ are parallel respectively to the tangents at B to δ, γ, and meet a point P on the radical axis of the circles. Then P and B are two of the harmonic points of the quadrangle ACDF, and the line joining P to the third harmonic point is parallel to the line through B on which the circles make equal intercepts.**

This follows at once from what has been proved, since the lines joining P to the other harmonic points from a pair harmonic with the pair PAC, PDF.

6. **Let two conics γ, δ meet in the points B, E, L, M, let the tangents at B to γ, δ meet LM in G, H, let any line through B meet γ, δ again in A, F and let EA cut BH in V and EF cut BG in U. Then UV and AF intersect on the line LM, while HA and GF intersect on the line BE.**

If EF, EA cut LM in R, S, what we have first to prove is that the ranges EASV, EFRU have the same cross ratio, and we prove this by projecting them both from B on the line LM, that is, we prove that if AF, BE meet LM in K, N, then the cross ratios NKSH, NKRG are the same.

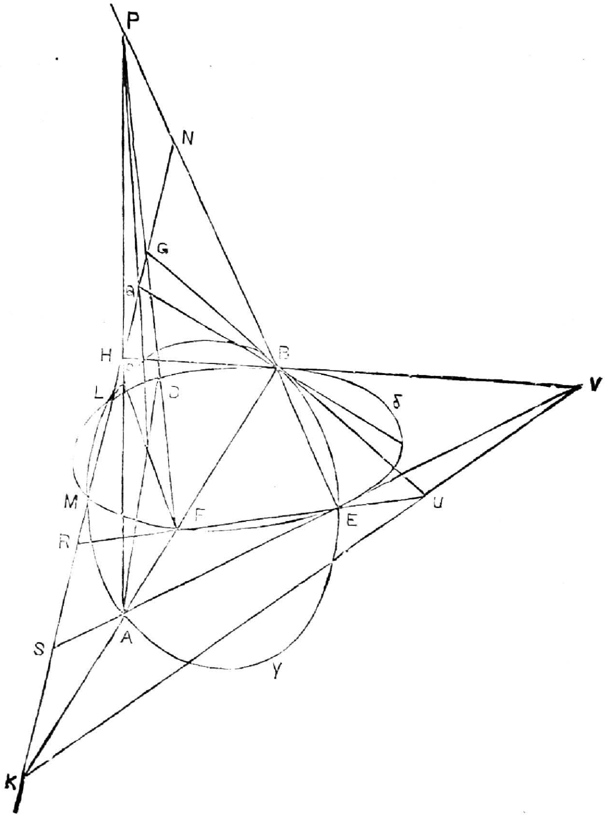
Using ABPQ to denote the cross ratio (AP/PB)÷(AQ/QB), we have ABPQ=ABRQ. ABPR. Now projection from the point E on to the conic γ will connect S and N with A and B, and projection from the point A on to the same conic will connect S and K with E and B, but we cannot connect by a single projection S with two of the three points N, K, H. To evaluate NKSH we must therefore express it as a product. We have

NKSH=NKSL. NKLH.

Now since

NLSK=NLSM. NLMK.
=E (BLAM). NLMK.
=B (GLAM). NLMK.
=GLKM. NLMK.
=LGMK. NLMK.
=NGMK.

we have also



NKSL = NKMG.

Similarly

NKLH = NKRM,

and therefore

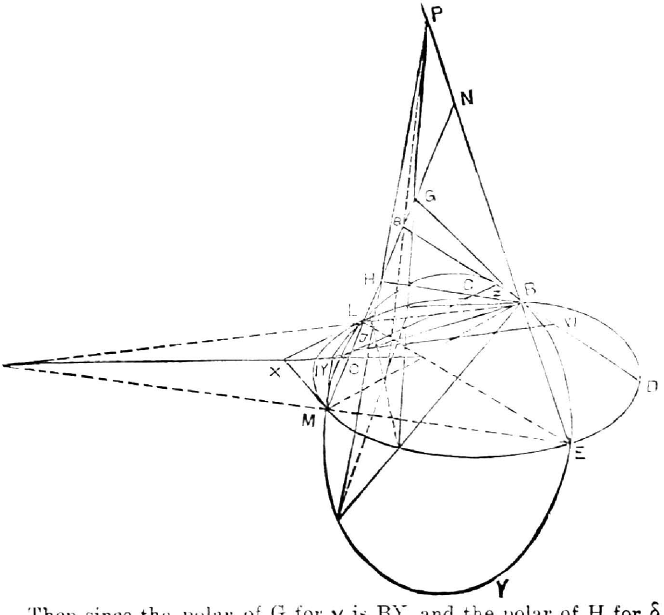
NKSH = NKRM.NKMG
= NKRG.

It follows that the ratios EASV, EFRU are equal. Hence UV FA meet on RS, that is, on LM, and also since HE and GE, HS and GR, HV and GU have common points lying on BE, the common point of HA and GF also lies on BE.

7. With the notation of 6, let Q be the point of LM forming with N a pair harmonic with the pair G, H and let P be any point of BE; then the quadrangle formed by the pair of points in which PH meets γ and the pair of points in which PG meets δ has P, B for two of its harmonic points and its third harmonic point lies in PQ.

8. With the same notation, the points distinct form B in which BQ meets the two conics form a pair harmonic with the pair B, Q.

Let J be the point in LM conjugate to Q, and O the point conjugate to N, with respect to the pair of points L, M and let Y, X be the poles of LM for the two conics γ, δ. The polar of N for all conics through B, E, L, M is the line joining the point where BL cuts EM with the points where BM cuts EL, and therefore Y, X, O all lie in this line; let BJ cut this line in I and let YJ, XJ cut QB in Z, W.



Then since the polar of G for γ is BY and the polar of H for δ is BX, the pencils B (LMGY) and B (LMHX) are harmonic; also by hypothesis the pencils B (LMQJ) and B (LMNO) are harmonic. Hence the pencil B (YXJO) has the same ratio as the pencil B (GHQN), that is to say, is a harmonic pencil, and therefore the range YXIO and the pencil J (YXIO) are both harmonic. Thus the pair of points Z, W is harmonic with respect to the pair Q, B.

Again, YJ is the polar of Q for γ and XW is the polar of Q for δ. Hence if BQ meets γ again in C and δ again in D, the ranges QZCB, QWDB are both harmonic, and since QBZW has been proved to be harmonic, it follows that QBCD also is harmonic, which is what we wish to establish.

To reach these conclusions analytically, we may take GHB for our triangle of reference, and suppose the conics γ, δ to have the equations

$$ax^2+2hny+by^2+2fyz=0,$$
$$ax^2+2hxy+by^2+2gxz=0,$$

where $ax^2+2hxy+by^2=0$ is the equation of the pair of lines BL, BM. The line BE has for its equation $gx=fy$ and so we may take for any point P in this line the co-ordinates f, g, k. Any point in PG has co-ordinates $f+m$, g, k, for some value of m, any point in PH has co-ordinates f, $g+n$, k, and the equation of the line joining them is

$$k(xn+ym)=(fn+gm+mn)z.$$

Now the first of these points lies on the conic δ if

$$a(f+m)^2+2h(f+m)g+bg^2+2g(f+m)k=0,$$

that is, if

$$a(f+m)^2+2(h+k)(f+m)2+bg^2=0,$$

and similarly the second lies on the conic γ is

$$af^2+2(h+k)f(g+n)+b(g+n)^2=0.$$

From the form of these two equations it follows that if m_1, m_2 are the roots of the first, we may take for the roots of the second n_1, n_2, where

$$(f+m_1)/f=g/(g+n_1),\ (f+m_2)/f=g/(g+n_2),$$

that is, where

$$fn_1+gm_1+m_1n_1=0,\ fn_2+gm_2+m_2n_2=0.$$

Then the lines joining the point $f+m_1$, g, k to the point f, $g+m_1$, k and the point $f+m_2$, g, k to the point f, $g+m_2$, k have equations

$$xn_1+ym_1=0,\ xn_2+ym_2=0,$$

and both pass through B, which is therefore one harmonic point of the quadrangle formed by these four points.

If the line joining P to the other harmonic point of the quadrangle meets GH in Q, while PB meets GH in N, then PQ, PN are harmonic with respect to PG, PH and so Q, N are harmonic with respect to G, H. But the co-ordinates of N may be taken as f,g,o and therefore the co-ordinates of Q may be taken as $f,-g,o$, and those of any point other than B in BQ as $f,-g,t$. The value of t for this point to lie on γ is given by

$$af^2-2\ hfg+bg^2-2fgt=0,$$

and the value for the point to lie on δ by

$$af^2-2h\ fg+bg^2+2fgt=0,$$

and since these values differ only in sign the two points are harmonic with respect to B, Q.

9th April 1914,
Trin. Coll., Cambridge

E. H. NEVILLE.

The Face of the Sky for September and October 1914.

The Sun

enters the autumnal equinox on September 24 at 3·5 A.M.

Phases of the Moon.

		September.			*October.*	
	D.	H.	M.	D.	H.	M.
Full Moon ...	4	7	31 P.M.	4	11	29 A.M
Last Quarter ...	12	11	18 ,,	12	3	3 P.M.
New Moon ...	20	3	3 A.M.	19	0	3 P.M.
First Quarter ...	26	5	33 P.M.	26	4	14 A.M.

There will be a partial eclipse of the Moon on September 4 at about 5·16 A.M. and lasting beyond the setting of the Moon.

Planets.

Mercury begins as an evening star. It is in conjunction with the Moon on September 21 and on October 21 and with Mars on October 30.

Venus is an evening star attaining its greatest elongation (46° 27′ E). It is in conjunction with the Moon on September 23 and October 22. It attains its greatest brilliancy on October 23.

Mars is in conjunction with the Moon on September 22.

Jupiter is in conjunction with the Moon on October 27.

Saturn is in quadrature to the Sun on September 26. It is stationary on October 15. It is in conjunction with the Moon on September 13 and October 11.

Uranus is stationary on October 18. It is in conjunction with the Moon on September 2, September 29 and October 26.

Neptune is in quadrature to the Sun on October 24. It is in conjunction with the Moon on September 16 and October 13.

V. RAMESAM.

SOLUTIONS.

Question 297.

(D. D. Kapadia, M.A., B.Sc.) :—Prove that the determinant of the nth order

$$\begin{vmatrix} a, & b, & b \;\dots & b, & b, & b \\ b, & a, & b \;\dots & b, & b, & b \\ a, & a, & b \;\dots & a, & a, & a \\ & \dots & \dots & \dots & \dots & \\ & \dots & \dots & \dots & \dots & \\ a & a & a \;\dots & b, & a, & a \\ b & b & b \;\dots & b, & a, & b \\ b & b & b \;\dots & b, & b, & a \end{vmatrix} = \{ (n-5)a-3b \} (b-a)^{n-4}$$

where in the first, second, last but one, and last rows there are $(n-1)$ b's and only one b in the other rows.

Solution by N. Sankara Aiyar, M.A.

Subtracting the last column from each of the others we get

$$\Delta = \begin{vmatrix} a-b, & 0, & 0 \;\dots & 0, & 0, & b \\ 0, & a-b, & 0 \;\dots & 0, & 0, & b \\ 0, & 0, & b-a, \;\dots & 0 & 0 & a \\ \dots & \dots & \dots & \dots & & \\ 0, & 0, & 0 \dots & b-a, & 0, & a \\ 0, & 0, & 0 \dots & 0, & a-b, & b \\ b-a, & b-a, & b-a \dots & b-a, & b-a, & a \end{vmatrix}$$

$$= (b-a)^{n-1} \begin{vmatrix} -1, & 0, & 0, \;\dots & 0, & 0, & b \\ 0, & -1, & 0, \;\dots & 0, & 0, & b \\ 0, & 0, & 1, \;\dots & 0, & 0, & a \\ & \dots & \dots & & \dots & \\ 0, & 0, & 0, \;\dots & 1, & 0, & a \\ 0, & 0, & 0, \;\dots & 0, & -1, & b \\ 1, & 1, & 1, \;\dots & 1, & 1, & a \end{vmatrix}$$

$$=(b-a)^{n-1}\begin{vmatrix} -1, & 0, & 0, \ldots 0, & 0, & 0 \\ 0, & -1, & 0, \ldots 0, & 0, & b \\ 0, & 0, & 1, \ldots 0, & 0, & a \\ & \ldots & \ldots & \ldots & \\ 0, & 0, & 0, \ldots 1, & 0, & a \\ 0, & 0, & 0, \ldots 0, & -1, & b \\ 1, & 1, & 1, \ldots 1, & 1, & a+b \end{vmatrix}$$

by multiplying the 1st column by b and adding to the last

$$=-(b-a)^{n-1}\begin{vmatrix} -1, & 0, \ldots & 0, & 0, & b \\ 0, & 1, \ldots & 0, & 0, & a \\ 0, & 0, \ldots & 0, & 0, & a \\ & \ldots & \ldots & \ldots & \\ 0, & 0, \ldots & 0, & -1, & b \\ 1, & 1, \ldots & 1, & 1, & a+b \end{vmatrix}$$

where the determinant is of the $(n-1)^{th}$ order.

Proceeding thus we get

$$\Delta=(b-a)^{n-1}\begin{vmatrix} 1, & 0, & 0, \ldots 0, & 0, & \\ 0, & 1, & 0 \ldots 0, & 0, & a \\ 0, & 0, & 1 \ldots 0, & 0, & a \\ & \ldots & \ldots & \ldots & \\ 0, & 0, & 0 \ldots 1, & 0, & \\ 0, & 0, & 0 \ldots 0, & -1, & \\ 1 & 1, & 1, & 1, & a+2b \end{vmatrix}$$

$$=(b-a)^{n-1}\begin{vmatrix} -1, & \\ 1, & 2b-(n-5)a \end{vmatrix}$$

$$=(b-a)^{n-1}\{(n-5)\,a-3b\}$$

The answer given is evidently incorrect as the determinant is of the nth order and the power of $(b-a)$ should therefore be $n-1$.

Question 329.

(M. Bhimasena Rau) :—If the pedal circle of P with respect to a triangle ABC touches the nine point circle of ABC, show that the sum of the angles PAB, PBC, PCA, is constant.

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Solution by N. Sankara Aiyar, M.A.

With the given condition the locus of P is given by

$$\Sigma x \cos A\,(y^2-z^2)=0,$$

(See p. 39, Vol. II, J.I.M.S.)

i.e., $\Sigma xy^2 \sin B \sin C=\Sigma(xy^2 \cos B \cos C+xz^2 \cos A)$.

Now $\cos(A+B+C)=-1$.

$\therefore\ \cos A \cos B \cos C+1=\Sigma \sin A \sin B \cos C$.

$\therefore\ \Sigma(xy^2 \sin B \sin C)+xyz\Sigma(\sin A \sin B \cos C)$
$=xyz(1+\cos A \cos B \cos C)+\Sigma(xy^2\cos B \cos C+xz^2 \cos A)$

$\therefore\ \Sigma xy \sin B \sin C\,(y+z \cos A)$
$=(x+y\cos C)(y+z \cos A)(z+x \cos B)$.

$$\therefore\quad \sum\frac{xy \sin B \sin C}{(x+y \cos C)(z+x \cos B)}=1.$$

If $PAB=\alpha$, $PBC=\beta$, $PCA=\gamma$, we get

$$\Sigma\tan\alpha \tan\beta=1.$$

$$\therefore\quad \alpha+\beta+\gamma=90^\circ.$$

Hence the sum of the angles in question is constant and equal to a right angle.

Question 465.

(S. NARAYANA AIYAR, M.A.) :—When n lies between -1, and -2, shew that

$$(1)\quad \int_0^\infty x^{n-1}\log\frac{\sinh x}{x}\,dx=\frac{\pi^{n+1}}{n\sin\frac{\pi n}{2}}\,S_n.$$

$$(2)\quad \int_0^\infty x^{n-1}\log\frac{\cosh x-\cos x}{x^2}\,dx=\frac{2^{\frac{n}{2}}\,\pi^{n+1}S_n}{n\sin\frac{\pi n}{4}}$$

where $S_n=1^n+2^n+3^n$..................to infinity.

Solution by N. Sankara Aiyar, M.A.

$$\text{Now}\quad \frac{\sinh x}{x}=\prod_{r=1}^{\infty}\left(\left(1+\frac{x^2}{r^2\pi^2}\right)\right).$$

$$\therefore\quad I=\int_0^\infty \Sigma x^{n-1}\log\left(1+\frac{x^2}{r^2\pi^2}\right)dx.$$

$$=\sum_{r=1}^{\infty}\left[\pi^n r^n\int_0^\infty y^{n-1}\log(1+y^2)\,dy.\right]$$

$$=\sum_{r=1}^{\infty}\left[\pi^n r^n\left\{\left[\frac{y^n \log(1+y^2)}{n}\right]_0^{\infty}-\frac{2}{n}\int_0^{\infty}\frac{y^{n+1}}{(1+y^2)}dy\right\}\right]$$

$$=\sum_{r=1}\left[-\pi^n r^n\frac{1}{n}\frac{2\pi}{2\cos\frac{(n+1)\pi}{2}}\right]$$

$$=\sum_{r=1}^{\infty}\frac{\pi^{n+1} r^n}{n\sin\frac{n\pi}{2}}$$

$$=\frac{\pi^{n+1}}{n\sin\frac{n\pi}{2}}S_n.$$

Again $\frac{\cosh x-\cos x}{x^2}=\prod_{r=1}^{\infty}\left(1+\frac{x^4}{4r^4\pi^4}\right).$

(Chrystal, Part II, p. 359.)

$$\therefore\ 1=\int_0^{\infty}\sum_{r=1}^{\infty}\left[x^{n-1}\log\left(1+\frac{x^4}{4r^4\pi^4}\right)dx\right]$$

$$=\sum_{r=1}^{r=\infty}\left[2^{\frac{n}{2}}r^n\pi^n\int_0^{\infty}y^{n-1}\log(1+y^4)\,dy\right]$$

$$=\sum_{r=1}^{r=\infty}\left[2^{\frac{n}{2}}r^n\pi^n\left\{\left[\frac{y^n}{n}\log(1+y^4)\right]_0^{\infty}-\frac{4}{n}\int_0^{\infty}\frac{y^{n+3}}{1+y^4}dy\right\}\right]$$

$$=\sum_{r=1}^{r=\infty}\left[2^{\frac{n}{2}}r^n\pi^n\frac{4\pi}{4n\sin\frac{n\pi}{4}}\right]=\frac{2^{\frac{n}{2}}n^{n+1}}{n\sin\frac{n\pi}{4}}S_n.$$

Question 510.

(S. P. Singaravelu Mudaliar) :—Show that the locus of the symmedian point of the triangle of maximum area inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{4}\left(\frac{a^2-b^2}{a^2+b^2}\right).$$

Solution by T. P. Trivedi, M.A., LL.B., N. Sankara Aiyar, M.A., and D. Krishnamurti.

Let θ, $\theta+\frac{2\pi}{3}$, $\theta+\frac{4\pi}{3}$ be the vertices of the angular points A, B, C of the maximum triangle ABC in the ellipse.

We have

$$BC^2=a^2\left\{\cos\left(\theta+\frac{2\pi}{3}\right)-\cos\left(\theta+\frac{4\pi}{3}\right)\right\}^2$$

$$+b^2\left\{\sin\left(\theta+\frac{2\pi}{3}\right)-\sin\left(\theta+\frac{4\pi}{3}\right)\right\}^2$$

$$=3a^2\sin^2\theta+3b^2\cos^2\theta.$$

Similarly $$CA^2=3a^2\sin^2\left(\theta+\frac{2\pi}{3}\right)+3b^2\cos^2\left(\theta+\frac{2\pi}{3}\right),$$

and $$AB^2=3a^2\sin^2\left(\theta+\frac{\pi}{3}\right)+3b^2\cos^2\left(\theta+\frac{\pi}{3}\right).$$

The areal co-ordinates of the symmedian point are proportional to BC^2, CA^2, AB^2; thus if (x, y) be the symmedian point we have

$$\frac{x}{a}=\frac{BC^2\cos\theta.+CA^2\cos\left(\theta+\frac{2\pi}{3}\right)+AB^2\cos\left(\theta+\frac{4\pi}{3}\right)}{BC^2+CA^2+AB^2}$$

and $$\frac{y}{b}=\frac{BC^2\sin\theta.+CA^2\sin\left(\theta+\frac{2\pi}{3}\right)+AB^2\sin\left(\theta+\frac{4\pi}{3}\right)}{BC^2+CA^2+AB^2}.$$

Now

$$BC^2+CA^2+AB^2=3a^2\left\{\sin^2\theta+\sin^2\left(\theta+\frac{\pi}{3}\right)+\sin^2\left(\theta+\frac{2\pi}{3}\right)\right\}$$

$$+3b^2\{\cos^2\theta+\ldots\ldots+\ldots\ldots\}$$

$$=\frac{9}{2}(a^2+b^2).$$

Again

$$\cos\theta\sin^2\theta+\cos\left(\theta+\frac{2\pi}{3}\right)\sin^2\left(\theta+\frac{2\pi}{3}\right)$$

$$+\cos\left(\theta+\frac{4\pi}{3}\right)\sin^2\left(\theta+\frac{\pi}{3}\right)$$

$$=\frac{1}{4}\left\{2\sin 2\theta\sin\theta+2\sin\left(2\theta+\frac{4\pi}{3}\right)\sin\left(\theta+\frac{2\pi}{3}\right)\right.$$

$$\left.-2\sin\left(2\theta+\frac{2\pi}{3}\right)\sin\left(\theta+\frac{\pi}{3}\right)\right\}$$

$$=\tfrac{1}{4}\left\{\cos\theta-\cos 3\theta+\cos\left(\theta+\frac{2\pi}{3}\right)-\cos(3\theta+2\pi)\right.$$
$$\left.-\cos\left(\theta+\frac{\pi}{3}\right)+\cos(3\theta+\pi)\right\}$$
$$=-\tfrac{3}{4}\cos 3\theta.$$

Similarly

$$\cos^3\theta+\cos^3\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)\cos^2\left(\theta+\frac{\pi}{3}\right)$$
$$=\cos^3\theta+\cos^3\left(\theta+\frac{2\pi}{2}\right)-\cos^3\left(\theta+\frac{\pi}{3}\right)$$
$$=\tfrac{3}{4}\cos 3\theta, \text{ on simplification.}$$

Thus
$$\frac{x}{a}=\frac{\frac{9}{4}\cos 3\theta(-a^2+b^2)}{\frac{9}{2}(a^2+b^2)}=-\frac{\cos 3\theta}{2}\,\frac{a^2-b^2}{a^2+b^2},$$

and
$$\frac{y}{b}=-\frac{\sin 3\theta}{2}\,\frac{a^2-b^2}{a^2+b^2}.$$

Hence the locus is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{4}\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

Question 511.

(S. Krishnaswami Aiyangar) :—If S, H are the foci of the maximum inscribed ellipse of the triangle of reference, prove that

$$AS.BS.CS.AH.BH.CH=\frac{a^2b^2c^2}{27}.$$

Solution by N. Sankara Aiyar, M. A.

Now $AS.BS.CS.AH.BH.CH=16\,R^2\alpha^2\beta^2$.

(see p. 143, Vol. IV., J. I.M.S.)

But
$$\alpha^2\beta^2=\frac{\Delta^2}{27} \text{ (see p. 113, Vol. V.)}$$

$$\therefore \quad AS.BS.CS.AH.BH.CH=\frac{16\,R^2\Delta^2}{27}=\frac{a^2b^2c^2}{27}.$$

Or thus

$$AS.AH=\frac{bc}{3} \text{ (see p. 14, Vol. VI., J.I.M.S.)}$$

$$\therefore \quad AS.BS.CS.AH.BH.CH=\frac{a^2b^2c^2}{27}.$$

Question 512.

(A. Narasinga Row) :—If $n\phi(n)+(n+2)\phi((n+2)=\phi(n+1)$, prove that

(i) $\phi(1)x-\phi(3)x^3+\phi(5)x^5\ldots\ldots=\phi(1)\sin(\tanh^{-1}x)$

(ii) $\phi(0)-\phi(2)x^2+\phi(4)x^4\ldots=\phi(0)+\phi(1)\{\cos(\tanh^{-1}x)-1\}$

Solution by N. Sankara Aiyar, M.A.

Let $y=\cos(\tanh^{-1}x)$

Differentiating $\quad y_1=-\dfrac{\sin(\tanh^{-1}x)}{1-x^2},$

and $\quad y_2(1-x^2)-2xy_1=-\dfrac{\cos(\tanh^{-1}x)}{1-x^2}.$

$\therefore \quad y_2(1-x^2)^2-2xy_1(1-x^2)=-y.$

i.e., $\quad y_2(1-2x^2+x^4)-2xy_1(1-x^2)+y=0.$

Put $y=\Sigma A_r x^r$. Then, substituting and equating coefficients in the above equation, we get

$$(r+2)(r+1)A_{r+2}-(2r^2-1)A_r+(r-1)(r-2)A_{r-2}=0 \quad \ldots \quad (1)$$

Now it is clear that the differential equation is satisfied by $\sin(\tanh^{-1}x)=y$ also.

Taking $y=\sin(\tanh^{-1}x)$, we see that y changes sign with x and is zero when x is zero; hence it is an odd function of x.

Again differentiating $\sin(\tanh^{-1}x)=\Sigma A_{2r+1}x^{2r+1}$, we get

$$\cos(\tanh^{-1}x)=(1-x^2)\Sigma(2^{r+1})\ A_{2r+1}x^{2r}.$$

$$\therefore \quad A_{2r}=(2r+1)A_{2r+1}-(2r-1)A_{2r-1} \ \ldots \quad \ldots \quad (2)$$

Substituting in equation (1), we get

$$A_{2r+1}=(2r+2)A_{2r+2}-2rA_{2r},$$

making use of the similar condition got by differentiating the value of $\cos(\tanh^{-1}x)$ and equating to $\sin(\tanh^{-1}x)$.

Hence if $\sin(\tanh^{-1}x)=\Sigma(-1)A_{2r+1}x^{2r+1}$, the equation connecting successive A's is

$$A_r=(r+1)A_{r+1}+(r-1)A_{r=1};$$

or putting $\phi(n)$ for A_n

$$\phi(n+1)=n\phi(n)+(n+2)\phi(n+2).$$

Similarly $\cos(\tanh^{-1}x)=A_1+\Sigma(-1)^r A_{2r}$.

Hence we get

$$\phi(1)\sin(\tanh^{-}x)=\phi(1)x-\phi(3)x^3+\phi(5)x^5+\ldots\ldots$$

and $\quad \phi(1)\{\cos(\tanh^{-1}x)-1\}+\phi(o)=\phi(o)-\phi(2)x+\ldots\ldots$

remembering that $A_1=1$.

Again put $x=iy$, then

$$\Sigma\phi(n)y^n=\phi(o)-\phi(1)+\phi(1)e^{-i\tanh^{-1}x}$$

$$=\phi(o)-\phi(1)+\phi(1)\,e^{-i\tanh^{-1}yi)}$$

$$=\phi(o)-\phi(1)+\phi(1)e^{\tan^{-1}y}.$$

In other words

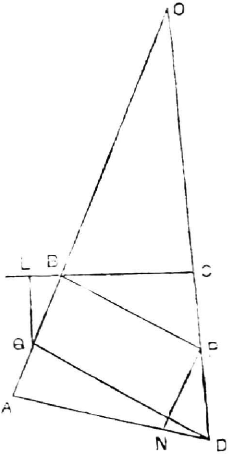
$$\Sigma\phi(n)x^n=\phi(o)-\phi(1)+\phi(1)\,e^{\tan^{-1}x}.$$

Hence the given series is always convergent.

Question 515.

(A. C. L. WILKINSON, M.A., F. R. A. S.):—ABCD is a quadrilateral. DQ, BP are any two parallel straight lines meeting AB, CD, respectively in Q and P. QL parallel to CD meets BC in L, and PN parallel to AB meets AD in N. Prove that the middle points of AC, DL, BN are collinear.

Analytical solution by D. Krishnamurti.



Take OA, OD as the co-ordinate axes where O is the point of intersection of AB, CD.

Let $OB=a$, $OC=b$, $CP=k$, $PN=c$. Then, if $BQ=\lambda\, a$, $PD=\lambda\,(b+k)$, since BP and QD are parallel.

Now the equation to DN is

$$\frac{x-c}{y-(b+k)}=-\frac{c}{\lambda(b+k)};$$

putting $y=o$, we get $OA=c+\frac{c}{\lambda}$.

Comparing Δ^s LBQ, CBO

$$QL=\lambda a.\frac{b}{a}=\lambda b.$$

Hence the co-ordinates of the following points are known

B (a, o) C (o, b) L $(a+a\lambda, -\lambda b)$

N $(c, b+k)$ A $\left(c+\frac{c}{\lambda}, o.\right)$ D $(o, \lambda+1\ b+k)$

The co-ordinates of the mid-points of BN, CA, LD are therefore

$$\left(\frac{a+c}{2}, \frac{b+k}{2}\right)\left(\frac{c\lambda+c}{2\lambda}; \frac{b}{2}\right)\left(\frac{a+a\lambda}{2}\ \frac{b+k+\lambda k}{2}\right)$$

If these are collinear then the usual determinant must vanish identically : multiplying by λ the column of the first determinant, we get

$$\Delta\equiv\begin{vmatrix}\lambda(a+c) & b+k & 1\\ c(\lambda+1) & b & 1\\ a\lambda(\lambda+1) & b+k+k\lambda & 1\end{vmatrix}$$

Subtracting the 1st row from each of the 2nd and 3rd, we get

$$\Delta \equiv -\lambda \begin{vmatrix} \lambda(a+c) & b+k & 1 \\ c-a\lambda & -k & 0 \\ (c-a\lambda) & -k & 0 \end{vmatrix} \equiv 0,$$

since two rows are identical.

Hence the mid-points of AC, DL, BN are collinear.

Question 516.

(K. J. Sanjana, M.A.) :—TP, TQ tangents to a conic of centre C and focus F, cut the auxiliary circle in Y, Z ; and FW is perpendicular to the chord of contact ; a tangent of the conic perpendicular to TC cuts FY, FZ, FW in y, z, w, respectively. Prove that $yw=wz$, and enunciate the corresponding property for the parabola.

Additional solution by T. P. Trivedi, M.A., LL. B.

Let TC be joined. Since TC bisects PQ, the lines TP, TC, TQ and the parallel from T to PQ from a harmonic pencil.

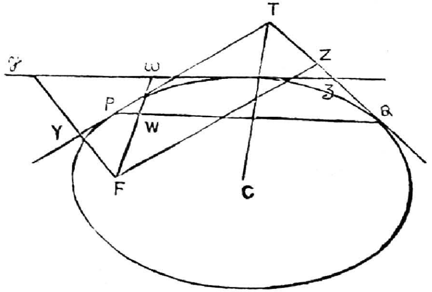
Join FY and FZ, which are perp. to TP and TQ respectively.

As FW is perpendicular to PQ, and as ywz is perp. to TC, a parallel from F to ywz will also be perp. to TC.

Thus FY, FW, FZ and a parallel from F to ywz from a harmonic pencil ; whence we infer $yw=wz$.

For the parabola this becomes :—

TP, TQ are two tangents to a parabola cutting the tangent at the vertex in Y and Z, and FW perpendicular to PQ meets the tangent at the vertex in W ; then YW=WZ.



QUESTIONS FOR SOLUTION.

562. (N. P. PANDYA) :—Solve the equation

$$2\left(5+9p^2+9y\frac{dp}{dx}\right)\frac{dp}{dx}+6(x+4py)\frac{d^2p}{dx^2}$$
$$+(x^2+3y^2)\frac{d^3p}{dx^3}=\frac{18}{x^4}-\frac{24x(1-x^2)}{(1+x^2)^4},$$

where $p=\frac{dy}{dx}$.

563. (N. P. PANDYA) :—Given two quadrilaterals in the plane of the paper shew how to draw a straight line bisecting them both.

564. (E. H. NEVILLE, B.A., B.SC.) :—If D denotes the determinant

$$\begin{vmatrix} 1 & m_1 & n_1 & a_1m_1 & a_1n_1 \\ 1 & m_2 & n_2 & a_2m_2 & a_2n_2 \\ 1 & m_3 & n_3 & a_3m_3 & a_3n_3 \\ 1 & m_4 & n_4 & a_4m_4 & a_4n_4 \\ 1 & m_5 & n_5 & a_5m_5 & a_5n_5 \end{vmatrix}$$

and D_r, M_r, N_r, P_r, Q_r are the co-factors of the elements of the rth row, prove that

$$\begin{vmatrix} \Sigma(a_rm_rk_r), & \Sigma(m_rk_r) \\ \Sigma(a_rn_rk_r), & \Sigma(n_rk_r) \end{vmatrix}=D\Sigma(k_r).$$

where k_r denotes $(M_rQ_r-N_rP_r)/D_r$ and the summations include the five values of r.

565. (S. KRISHNASWAMI AIYANGAR):—Shew that the locus of the orthopoles of tangents to the maximum inscribed ellipse of a triangle is a straight line passing through the orthocentre of the triangle.

566. (S. KRISHNASWAMY AIYANGAR):—Prove that

$$\sin^{-1}x=\frac{x}{(1-x^2)^{\frac{3}{2}}}\left\{1-s_1\frac{x^2}{1-x^2}+s_3\left(\frac{x^2}{1-x^2}\right)^2-\ldots\ldots\right\}$$

where s_r = the sum of the reciprocals of the first r odd numbers.

567. (P. A. SUBRAMANYA AIYAR, B.A., L.T.):—Shew that a uniform tetrahedron of mass m is equi-momental with six particles of mass $\frac{1}{10}m$ at the middle points of its six edges and one of mass $\frac{2}{5}m$ at the centre of mass of the tetrahedron.

568. (K. V. ANANTHANARAYANA SASTRI, B.A.):—Evaluate

$$\int_0^{\infty}\frac{\log x\,dx}{1+x^4}.$$

569. (R. VYTHYNATHASWAMY):—If p points are chosen at random on the circumference of a circle, find the chance of their lying on an arc equal to a kth part of the circumference.

570. (Professor SANJANA) :—Prove that when n and k are positive integers the sums of the following series vanish—

$$(a)\ 1-\binom{n+2k}{2}\frac{1}{2n+2k-1}+\binom{n+2k}{4}\frac{1.3}{(2n+2k-1)(2n+2k-3)}$$
$$-\binom{n+2k}{6}\frac{1\cdot3\cdot5}{(2n+2k-1)(2n+2k-3)(2n+2k-5)}+\dots\dots\dots,$$
$$(b)\ \binom{2n+2k+1}{1}-\binom{2n+2k+1}{3}\frac{2n+3}{2n+2k-1}$$
$$+\binom{2n+2k-1}{5}\frac{(2n+3)(2n+5)}{(2n+2k-1)(2n+2k-3)}-\dots\dots$$

where $\binom{x}{y}$ denotes ${}_xC_y$

571. (S. RAMANUJAN) :—If $\frac{\pi\alpha}{2}=\log\tan\left(\frac{\pi}{4}+\frac{\pi\beta}{4}\right)$, shew that

$$\left(\frac{1^2+\alpha^2}{1^2-\beta^2}\right)\left(\frac{3^2-\beta^2}{3^2+\alpha^2}\right)^3\left(\frac{5^2+\alpha^2}{5^2-\beta^2}\right)^5\dots\dots\dots=e^{\frac{1}{2}\pi\alpha\beta}$$

572. (M. BHIMASENA RAO) :—If $f(x)=x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\frac{x^7}{7^2}+\dots\dots$, shew that

$$\text{(i)}\ f(\sqrt{2}-1)=\frac{\pi^2}{16}-\tfrac{1}{4}\{\log(\sqrt{2}-1)\}^2$$
$$\text{(ii)}\ f\left(\frac{\sqrt{5}-1}{2}\right)=\frac{\pi^2}{16}-\frac{3}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$
$$\text{(iii)}\ f(\sqrt{5}-2)=\frac{\pi^2}{24}-\frac{2}{4}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2$$

573. (M. BHIMASENA RAO) :—Shew that

$$\frac{1}{2}-\frac{2}{3^2}\cdot\frac{1}{2^3}+\frac{2\cdot4}{3\cdot5^2}\cdot\frac{1}{2^5}-\frac{2\cdot4\cdot6}{3\cdot5\cdot7^2}\cdot\frac{1}{2^7}+\dots\dots=\frac{\pi^2}{12}-\frac{3}{2}\left\{\log\frac{\sqrt{5}-1}{2}\right\}^2.$$

574. (V. RAMASWAMI AIYAR, M.A.) :—Given a tetrahedron ABCD and a line LL′. Prove that the pedal spheres, with respect to the tatrahedron, of points lying in LL′ cut a fixed sphere, Σ, orthogonally. Defining the centre of the sphere Σ as the *orthopole* of LL′, and denoting it by σ, show that, in general, there are three points in LL′ such that the feet of the perpendiculars from any of them on the faces of ABCD are coplanar, and that the orthopole σ is the point of intersection of the pedal planes of these points. Show also that the locus of the orthopoles of a system of parallel lines (LL′) is a plane, Γ, at right angles to the system.